MATH2050C Selected Solution to Assignment 9

Section 5.1 no. 3, 4ac, 5, 8, 10, 13.

(4a) The function $f(x) = [x]$ is continuous except at all integers.

(4c) The function $h(x) = [\sin x]$ is continuous whenever $\sin x$ is not equal to $-1, 0, 1$. At $x = 0$, $[\sin x] = 0$ for small $x > 0$ but $[\sin x] = -1$ for small $x < 0$, so it is not continuous at 0. Similarly, it is not continuous at all $n\pi$. On the other hand, $\sin x = 1$ if and only if $x = (2n+1/2)\pi, n \in \mathbb{Z}$. For x comes close to $(2n+1/2)\pi$ from its right or left, $\sin x$ is very close to 1 but less than 1, so $[\sin x] = 0$. As $[\sin \pi/2] = 1$, h is discontinuous at $(2n+1/2)\pi$. On the other hand, when x comes close to $3\pi/2$, $\sin x$ is greater and close to -1, hence $[\sin x] = -1 = [\sin 3\pi/2]$. Hence $[\sin x]$ is continuous at $(2n+3/2)\pi$. Conclusion: The discontinuity points of h are $n\pi$ and $(2n+1/2)\pi, n \in \mathbb{Z}$.

(5) We have

$$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2}(x+3) = 5 \ .$$

Therefore, the function $F(x) = f(x)$ when $x \neq 2$ and $F(2) = 5$ is a continuous function which extends f.

(8) Yes. Pick a sequence of rational numbers $\{r_n\}$ to tend to a given irrational number x. By continuity, $f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = g(x)$.

(13) Let x_0 be a continuity point of g. Let $\{x_n\}$ be a sequence of rational numbers tending to x_0. By continuity at x_0, $g(x_0) = \lim_{n\to\infty} g(x_n) = \lim_{x\to x_0} 2x_n = 2x_0$. On the other hand, let $\{y_n\}$ be an irrational sequence tending to x_0. We have $g(x_0) = \lim_{n\to\infty} g(y_n) = \lim_{n\to\infty}(y_n+3) = x_0+3$. We get $2x_0 = x_0 + 3$ which implies $x_0 = 3$. Conclusion: 3 is the unique continuity point for g.

Section 5.2 no. 1bc, 3, 7, 10, 11, 15.

(1b) g is continuous on $[0,\infty)$. For, both x and $\sqrt{x}$ are continuous functions on $[0,\infty)$, so is their sum $x + \sqrt{x} \in [0,\infty)$. As the function $y \mapsto \sqrt{y}$ is continuous on $[0,\infty)$, the composite function $g(x) = \sqrt{x+\sqrt{x}}$ is continuous on $[0,\infty)$.

(1c) $\sin x$ and the absolute value (function) are continuous on $(-\infty,\infty)$, so is their composite $|\sin x|$. It follows that $\sqrt{1+|\sin x|}$ (the composite of $1+|\sin x|$ and the square root function) is continuous on $(-\infty,\infty)$. As the quotient of two continuous functions is continuous away from where the denominator vanishes, we conclude that h is continuous on $(-\infty,0)\cup(0,\infty)$.

(7) Just let $f(x) = 1$ at rational x and $f(x) = -1$ at irrational x.

(12) Let $x_1 \in \mathbb{R}$. We would like to show f is continuous at x_1. Let $\{y_n\}$ be $y_n \to x_1$. Then $y_n - (x_1 - x_0) \to x_0$, so by additivity of f and continuity at x_0 we have $f(y_n) = f(y_n - (x_1 - x_0)) + f(x_1 - x_0) \to f(x_0) + f(x_1 - x_0) = f(x_1)$, done.

(13) f being additive means $f(x+y) = f(x)+f(y)$. By induction, $f(x_1+\cdots+x_n) = f(x_1)+\cdots+$

$f(x_n)$. Taking $x_1 = \cdots = x_n = x$, $f(nx) = nf(x)$. It follows that $f(1) = f(m1/m) = mf(1/m)$ which implies $f(1/m) = f(1)/m = c/m$. Then $f(n/m) = nf(1/m) = cn/m$, that is, $f(r) = cr$ for all rational numbers r. By continuity, $f(x) = cx$ for all x.

(15) The formula

$$\sup\{a, b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b| \ ,$$

can be verified by considering the cases $a < b$ and $a > b$ separately. Hence

$$h(x) = \sup\{f(x), g(x)\} = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)|$$

shows that h is continuous whenever f and g are continuous.

Note. Can you find a corresponding formula for $\inf\{f, g\}$?

Supplementary Problems

1. Determine the largest domain on which the function is defined and study its continuity.

   (a) $\sin x/x$.

   Solution This function is well-defined whenever $x \neq 0$. Hence its largest domain of definition is $(-\infty, 0) \cup (0, \infty)$. Since both x and $\sin x$ are continuous everywhere, by the quotient rule $\sin x/x$ is continuous on $(-\infty, 0) \cup (0, \infty)$. By the way, as we know $\sin x/x \to 1$ as $x \to 0$. One may extend this function to a new function $f(x) = \sin x/x, x \neq 0$ and $f(0) = 1$ which is continuous on the entire $\mathbb{R}$.

   (b) $\sqrt{\dfrac{x+6}{x+1}}$.

   Solution $\dfrac{x+6}{x+1} \in [0, \infty)$ if and only if $x > -1$ or $x \leq -6$. Hence it is well-defined on $(-\infty, -6] \cup (-1, \infty)$. As the square root function is continuous on $[0, \infty)$, by the composition rule $\sqrt{\dfrac{x+6}{x+1}}$ is continuous on $(-\infty, -6] \cup (-1, \infty)$.

   (c) $\mathrm{sgn}(x^2 - x - 2)$.

   Solution Let $f(x) = \mathrm{sgn}(x^2 - x - 2)$. $x^2 - x - 2 = (x-2)(x+1) = 0$ if and only if $x = 2, -1$. By the composition rule, this function is continuous at all x not equal to 2 or -1. On the other hand, we have $\lim_{x\to 2^+} f(x) = 1$ and $\lim_{x\to 2^-} f(x) = -1$. Therefore, f is not continuous at 2. Similarly, it is not continuous at -1.

   (d) $e^{1/\sin x}$.

   Solution The function is well-defined whenever $\sin x \neq 0$, hence its largest domain of definition is $\{x : x \neq n\pi, n \in \mathbb{Z}\}$. By the composition rule, it is continuous on this domain.

2. Show that the function

$$f(x) = \frac{(1+x)^{1/2} - 1}{(1+x)^{1/3} - 1}$$

   can be extended to be continuous at $x = 0$.

Solution Using

$$(1+x)^{1/2}-1=\frac{x}{\sqrt{1+x}+1}\ , \quad (1+x)^{1/3}-1=\frac{x}{(1+x)^{2/3}+(1+x)^{1/3}+1}\ ,$$

we have, for $x \neq 0$,

$$f(x)=\frac{(1+x)^{1/2}-1}{(1+x)^{1/3}-1}=\frac{(1+x)^{2/3}+(1+x)^{1/3}+1}{\sqrt{1+x}+1}\to\frac{3}{2}\ , \quad \text{as } x\to 0\ .$$

Hence by defining $f(0)=3/2$, f becomes continuous at 0.

3. Let f be defined in A. Suppose f is continuous at $c \in A$ and $f(c)>0$. Show that there is some $\delta>0$ such that $f(x)>0$ for $x \in A, |x-c|<\delta$.

   Solution Taking $\varepsilon = f(c)/2$, there is $\delta > 0$ such that $|f(x)-f(c)| < f(c)/2$ for $x \in A, |x-c|<\delta$. From $-f(c)/2 < f(x)-f(c) < f(c)/2$ we deduce $f(x) > f(c)-f(c)/2 = f(c)/2 > 0$, done.